Inference for a Progressively First-Failure Censored Competing Risks Data from the Kumaraswamy Distribution

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Abstract— In medical studies or in reliability analysis, it is quite common that the failure of any individual or any item may be attributable to more than one cause. Moreover, the observed data are progressively first-failure censored competing risks data, when the lifetime distributions are Kumaraswamy. This type of censoring contains as special cases various types of censoring schemes used in the literature. Based on this type of censoring, we derive the maximum likelihood estimators (MLE) of the unknown parameters and asymptotic confidence intervals It is assumed that the latent cause of failures have independent Kumaraswamy distributions with the common shape parameter α, but different shape parameters β_1 and β_2 When the common shape parameter α is known, the Bayes estimates of the another shape parameters have closed form expressions and the corresponding credible intervals also can be constructed explicitly. As expected, when the common shape parameter is unknown the explicit expressions of the Bayes estimators cannot be obtained. Hence, we propose Markov chain Monte Carlo (MCMC) method to compute the Bayes estimates and construct the credible intervals of the unknown parameters. One set of real data has been analyzed for illustrative purposes. Finally, we provide a Monte Carlo simulation to compare and select optimal censoring schemes.

Index Terms— Kumaraswamy distribution; Progressive first-failure-censoring; Competing risks; Maximum likelihood method; MCMC method; credible intervals.

1 INTRODUCTION

In reliability, medical or biological studies it is quite common that more than one cause of failure may be present at the same time. An investigator is often interested in the assessment of a specific cause in the presence of other causes . In the statistical literature this problem is known as the competing risks model. A lifetime experiment with $\tau = 2$ different risk factors competing for the failure of the experimental units is considered. The data for such a competing risks model consist of the lifetime of the failed item and an indicator variable which denotes the cause of failure. In analyzing the competing risks model, it is assumed that data consist of a failure time and an indicator denoting the cause of failure. Severals studies have been carried out under this assumption for both the parametric and the non-parametric setups, namely the exponential, lognormal, gamma, Weibull, generalized exponential or exponentiated Weibull; see for example [1], [2] , [3] , [4] , [5] ,[6] and [7]. Recently, the competing risks model has received considerable interest among the statisticians. See for example, [8] and [9].

E-mail: a_a_mod@yahoo.com Censoring occurs when exact lifetimes are known only for a portion of the individuals or units under study, while for the remainder of the lifetimes information on them is partial. There are several types of censored tests. The most common censoring schemes are Type-I (time) censoring, where the life testing experiment will be terminated at a prescribed time T, and Type-II (failure) censoring, where the life testing experiment will be terminated upon the r -th (r is pre-fixed) failure. However, the conventional Type-I and Type-II censoring schemes do not have the flexibility of allowing removal of units at points other than the terminal point of the experiment, also when the lifetimes of products are very high, the experimental time of a Type II censoring life test can be still too long. Because of these lack of flexibilities, Johnson [10] described a life test in which the experimenter might decide to group the test units into several sets, each as an assembly of test units, and then run all the test units simultaneously until occurrence the first failure in each group. Such a censoring scheme is called first-failure censoring. If an experimenter desires to remove some sets of test units before observing the first-failures in these sets this life test plan is called a progressive first-failure-censoring scheme which recently introduced by [11]. Readers may refer to [12], [13], [14], [15] and [16] for extensive reviews of the literature on progressive censoring.

The main aim of this paper is to develop a confidence interval and the MLE for the Kumaraswamy distribution based on the progressively first-failure-censored sample in the presence of competing risks. Therefore, the organization of the paper is as follows. In Section 2, we introduce the model and present the notation used throughout this paper. The maximum

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likelihood estimates of the unknown parameters are discussed, we also present the asymptotic confidence intervals in Section 3. In Section 4, firstly, we discussed Bayes estimation of the unknown parameters when the common shape parameter α is known and construction of credible intervals. Secondly the Bayes estimation of the unknown parameters when the common shape parameter α is unknown and construction of credible intervals using MCMC method. A real data set from [17] is analyzed in Section 5. In section 6, the different methods are compared using Monte Carlo simulations. Some concluding remarks are finally made in Section 7.

2 MODEL ASSUMPTIONS AND NOTATION

Before proceeding any further, we describe different notations we are going to use in this paper.

- X_i lifetime of the i -th unit.
- X_{ij} lifetime of the i -th individual under cause j , $j=1,2$.
- $F(.)$ cumulative distribution function (*cdf*) of X_i .
- f(.) probability density function (*pdf*) of $F(.)$.
- $F_j(.)$ *cdf of* X_{ij} .
- $f_j(.)$ *pdf* of $F_j(.)$.
- $S_j(.)$ survival function of X_{ij} .

$\delta_i(.)$ indicator variable denoting the cause of failure of the i -th individual.

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The model studied in the paper satisfies the following a sumptions

i) When n independent groups with k items within each group are put on a life test, R_1 groups and the group in which the first failure is observed are randomly removed from the test as soon as the first failure (say $X^{\mathbf{R}}_{1:m:n:k}$ and $\delta_1 \in \{1,2\}$) has occurred, R_2 groups and the group in which the second first failure is observed are randomly removed from the test when the second failure (say $X^{\mathbf{R}}_{2:m:n:k}$ and $\delta_2 \in \{1,2\}$) has occurred, and finally R_m $(m \le n)$ groups and the group in which the m -th first failure is observed are randomly removed from the test as soon as the m -th failure (say $X^{\mathbf{R}}_{m:m:n:k}$ and $\delta_m \in \{1,2\}$) has occurred. The $(X^{\mathbf{R}}_{1:m:n:k}, \delta_1) <$ $(X^{\mathbf{R}}_{2:m:n:k}, \delta_2) < \ldots <$ $(X^{\mathbf{R}}_{m:m:n:k}, \delta_m)$ are called progressively first-failure-censored competing risks order statistics with the progressive censoring scheme $\mathbf{R}=(R_1,R_2,...,R_m)$ and for each i , δ_i takes a value either 1 and 2 the causes of failures. It is clear that $n=m+R_1+R_2+...+R_m$. To simplify the notation we will use henceforth x_i instead of $X_{i:m:n:k}$, $i=1,2+...+m$.

ii) The lifetime of unit is denoted as X_i , $i=1,2+...+m$. The time at which the unit i fails due to cause j is X_{ij} , $j=1,2$. That is, $X_i = \min\{X_{i1}, X_{i2}\}$.

iii) The distribution of the random variable X_{ij} is Kumaraswamy distribution with shape parameters α and β_j , $j=1,2$ and $i=1,2+...+m$. That is, the (*pdf*) and (*cdf*) of X_{ij} , $j=1,2$, for each $i=1,2+...+m$, are

$$f_j(x) = \alpha\beta_j x^{\alpha-1}(1-x^{\alpha})^{(\beta_j-1)}, \ 0<x<1, \ (\alpha>0, \ \beta_j>0), \tag{1}$$

$$F_j(x) = 1-(1-x^{\alpha})^{\beta_j}, \ 0<x<1. \tag{2}$$

The corresponding reliability and failure rate functions of this distribution at some t , are given, respectively by

$$S_j(t) = (1-t^{\alpha})^{\beta_j}, \ 0<t<1, \tag{3}$$

and

$$H_j(t) = \alpha\beta_j t^{\alpha-1}(1-t^{\alpha})^{-1}, \ 0<t<1. \tag{4}$$

iv) The two-parameters Kumaraswamy distribution is unimodal for $\alpha>1$ and $\beta>1$, uniantimodal for $\alpha<1$ and $\beta<1$, increasing for $\alpha>1$ and $\beta\le1$, decreasing for $\alpha\le1$ and $\beta>1$ and constant for $\alpha=\beta=1$. Jones [18] investigated properties of the Kumaraswamy distribution and also some similarities differences between the beta and Kumaraswamy distributions. The Kumaraswamy distribution is applicable to a number of hydrological problems and many natural phenomena whose process values are bounded on both sides. In hydrology and related areas, the Kumaraswamy distribution has received considerable interest, see Cordeiro et al. (2010, 2012).

Based on the above assumptions, the available data is a progressively first-failure-censored competing risks sample which contains the following: $(X_{1:m:n:k}, \delta_1, R_1)$, $(X_{2:m:n:k}, \delta_2, R_2), \cdots, (X_{m:m:n:k}, \delta_m, R_m)$, where $X_{1:m:n:k} < X_{2:m:n:k} < \cdots < X_{m:m:n:k}$ denote the m observed failure times, δ_1, δ_2, $\cdots$, δ_m denote the causes of failures, and R_1 , R_2 , $\cdots$, R_m denote the number of groups removed from the test at the failure times $X_{1:m:n:k} < X_{2:m:n:k} < \cdots < X_{m:m:n:k}$. If the failure times of the $n \times k$ items originally in the test are from a continuous population with (*cdf*) $F(x)$ and (*pdf*) $f(x)$, the joint probability density function for $X^{\mathbf{R}}_{1:m:n:k}$, $X^{\mathbf{R}}_{2:m:n:k}, ...,$ $X^{\mathbf{R}}_{m:m:n:k}$ is given by [11].

3 ESTIMATION OF THE PARAMETERS

In this section, we first estimate the parameters α and β_j, $j=1,2$ by considering the maximum likelihood (ML) methods, and then we compute the observed Fisher information based on the likelihood equations. These will enable us to develop pivotal quantities based on the limiting normal distribution, the resulting pivotal quantities can be used to develop approximate confidence interval for the parameters.

3.1 Maximum Likelihood Estimation

Based on the observed sample $(X_{1:m:n:k},\delta_1,R_1)$, $(X_{2:m:n:k},\delta_2,R_2),\cdots,(X_{m:m:n:k},\delta_m,R_m)$, the likelihood function is;

$$\ell(\underline{x};\alpha,\beta_1,\beta_2)\propto\alpha^m\beta_1^{m_1}\beta_2^{m_2}\prod_{i=1}^{m}\frac{x_i^{\alpha-1}}{(1-x_i^{\alpha})}\times\exp[k(\beta_1+\beta_2)\sum_{i=1}^{m}(R_i+1)\log(1-x_i^{\alpha})]. \quad (5)$$

The log-likelihood function without the additive constant can be written as follows;

$$L(\underline{x};\alpha,\beta_1,\beta_2)=m\log\alpha+m_1\log\beta_1+m_2\log\beta_2+(\alpha-1)\sum_{i=1}^{m}\log(x_i)-\sum_{i=1}^{m}\log(1-x_i^{\alpha})+k(\beta_1+\beta_2)\sum_{i=1}^{m}(R_i+1)\log(1-x_i^{\alpha}). \quad (6)$$

Upon differentiating (6) with respect to α,β_1 and β_2, and equating each result to zero, three equations must be simultaneously satisfied to obtain MLEs of the parameters α,β_1 and β_2. Then, we have

$$\frac{\partial L(\underline{x};\alpha,\beta_1,\beta_2)}{\partial\alpha}=\frac{m}{\alpha}+\sum_{i=1}^{m}\log(x_i)+\sum_{i=1}^{m}\frac{x_i^{\alpha}\log(x_i)}{(1-x_i^{\alpha})}-(\beta_1+\beta_2)\sum_{i=1}^{m}\frac{k(R_i+1)x_i^{\alpha}\log(x_i)}{(1-x_i^{\alpha})}=0, \quad (7)$$

$$\frac{\partial L(\underline{x};\alpha,\beta_1,\beta_2)}{\partial\beta_1}=\frac{m_1}{\beta_1}+k\sum_{i=1}^{m}(R_i+1)\log(1-x_i^{\alpha})=0, \quad (8)$$

and

$$\frac{\partial L(\underline{x};\alpha,\beta_1,\beta_2)}{\partial\beta_2}=\frac{m_2}{\beta_2}+k\sum_{i=1}^{m}(R_i+1)\log(1-x_i^{\alpha})=0. \quad (9)$$

Thus, the MLE $\hat{\alpha}$, $\hat{\beta}_1$ and $\hat{\beta}_2$ of the parameter α, β_1 and β_2 can be obtained by solving the nonlinear likelihood Eqs. (7-9) using, for example, the Newton-Raphson iteration scheme.

Notice that both m_1 and m_2 follow binomial distributions with sample size m. Hence, $m_j\sim Bin(m,\beta_{3-i}/(\beta_1+\beta_2)),\ j=1,2$.

3.2 Approximate interval estimation

The asymptotic normal distribution for the MLEs can be obtained in the usual way. From the log-likelihood function in (6), we have

$$\frac{\partial^2 L(\alpha,\beta_1,\beta_2)}{\partial\alpha^2}=-\frac{m}{\alpha^2}+\frac{x_i^{\alpha}\log^2(x_i)}{(1-x_i^{\alpha})^2}-k(\beta_1+\beta_2)\sum_{i=1}^{m}\frac{(R_i+1)x_i^{\alpha}\log^2(x_i)}{(1-x_i^{\alpha})^2}, \quad (10)$$

$$\frac{\partial^2 L(\alpha,\beta_1,\beta_2)}{\partial\alpha\partial\beta_1}=\frac{\partial^2 L(\alpha,\beta_1,\beta_2)}{\partial\beta_1\partial\alpha}=-k\sum_{i=1}^{m}\frac{(R_i+1)x_i^{\alpha}\log(x_i)}{(1+x_i^{\alpha})^2}, \quad (11)$$

$$\frac{\partial^2 L(\alpha,\beta_1,\beta_2)}{\partial\alpha\partial\beta_2}=\frac{\partial^2 L(\alpha,\beta_1,\beta_2)}{\partial\beta_2\partial\alpha}=-k\sum_{i=1}^{m}\frac{(R_i+1)x_i^{\alpha}\log(x_i)}{(1+x_i^{\alpha})^2}, \quad (12)$$

$$\frac{\partial^2 L(\alpha,\beta_1,\beta_2)}{\partial\beta_1\partial\beta_2}=\frac{\partial^2 L(\alpha,\beta_1,\beta_2)}{\partial\beta_2\partial\beta_1}=0, \quad (13)$$

$$\frac{\partial^2 L(\alpha,\beta_1,\beta_2)}{\partial\beta_1^2}=-\frac{m_1}{\beta_1^2}, \quad (14)$$

and

$$\frac{\partial^2 L(\alpha,\beta_1,\beta_2)}{\partial\beta_2^2}=-\frac{m_2}{\beta_2^2}. \quad (15)$$

The Fisher information matrix $I(\alpha,\beta_1,\beta_2)$ is then obtained by taking expectations of minus Eqs. (10-15). Under some mild regularity conditions, $(\hat{\alpha},\hat{\beta}_1,\hat{\beta}_2)$ is approximately bivariately normal with mean (α,β_1,β_2) and covariance matrix $I^{-1}(\alpha,\beta_1,\beta_2)$. In practice, we usually estimate $I^{-1}(\alpha,\beta_1,\beta_2)$ by $I^{-1}(\hat{\alpha},\hat{\beta}_1,\hat{\beta}_2)$. A simpler and equally valid procedure is to use the approximation

$(\hat{\alpha},\hat{\beta}_1,\hat{\beta}_2)\sim N\left((\alpha,\beta_1,\beta_2),I_0^{-1}(\hat{\alpha},\hat{\beta}_1,\hat{\beta}_2)\right)$, wher $I_0^{-1}(\hat{\alpha},\hat{\beta}_1,\hat{\beta}_2)$ is the observed information matrix

$$I_0(\hat{\alpha},\hat{\beta}_1,\hat{\beta}_2)=-\begin{bmatrix}-\frac{\partial^2 L(\alpha,\beta_1,\beta_2)}{\partial\alpha^2} & -\frac{\partial^2 L(\alpha,\beta_1,\beta_2)}{\partial\alpha\partial\beta_1} & -\frac{\partial^2 L(\alpha,\beta_1,\beta_2)}{\partial\alpha\partial\beta_2}\\ -\frac{\partial^2 L(\alpha,\beta_1,\beta_2)}{\partial\beta_1\partial\alpha} & -\frac{\partial^2 L(\alpha,\beta_1,\beta_2)}{\partial\beta_1^2} & -\frac{\partial^2 L(\alpha,\beta_1,\beta_2)}{\partial\beta_1\partial\beta_2}\\ -\frac{\partial^2 L(\alpha,\beta_1,\beta_2)}{\partial\beta_2\partial\alpha} & -\frac{\partial^2 L(\alpha,\beta_1,\beta_2)}{\partial\beta_2\partial\beta_1} & -\frac{\partial^2 L(\alpha,\beta_1,\beta_2)}{\partial\beta_2^2}\end{bmatrix}.$$

Approximate confidence intervals for α,β_1 and β_2 can be found by taking $(\hat{\alpha},\hat{\beta}_1,\hat{\beta}_2)$ to be bivariately normally distributed with mean (α,β_1,β_2) and covariance matrix $I_0^{-1}(\hat{\alpha},\hat{\beta}_1,\hat{\beta}_2)$. Thus, the $100(1-\gamma)\%$ approximate confidence intervals for α,β_1 and β_2 are respectively, given by

$\hat{\alpha}\pm Z_{\gamma/2}\sqrt{v_{11}},\quad \hat{\beta}_j\pm Z_{\gamma/2}\sqrt{v_{ss}},\ s=2,3,$

where $v_{ss},s=1,2,3$ are the elements on the main diagonal of the covariance matrix $I_0^{-1}(\hat{\alpha},\hat{\beta}_1,\hat{\beta}_2)$ and $z_{\frac{\gamma}{2}}$ is the percentile of the standard normal distribution with right-tail probability $\gamma/2$.

4 Bayesian Estimations

In this section, we provide the Bayes estimates of the unknown parameters and the corresponding credible intervals, when the common shape parameter α is known and when it is unknown. Based on the observed sample

$(x_{1:m:n}, \delta_1, R_1)$, $(x_{2:m:n}, \delta_2, R_2)$, $\cdots$,$(x_{m:m:n}, \delta_m, R_m)$, the likelihood function (5) , is given by

$$\ell(\alpha,\beta_1,\beta_2 \mid \underline{x}) \propto \alpha^m \beta_1^{m_1} \beta_2^{m_2} \eta_1(\alpha,\underline{x}) \exp\left[k(\beta_1+\beta_2)\eta_2(\underline{x},\alpha)\right], \quad (16)$$

where

$$\eta_1(\alpha,\underline{x}) = \prod_{i=1}^{m} \frac{x_i^{\alpha-1}}{(1-x_i^{\alpha})} \text{ and } \eta_2(\alpha,\underline{x}) = \sum_{i=1}^{m} \left(R_i+1\right) \log(1-x_i^{\alpha}). \quad (17)$$

4.1 Bayesian Estimations When α is Known

When the shape parameter α is known, the other shape parameters β_1 and β_2 have a conjugate gamma priors. It is assumed that the priors distribution of β_1 and β_2 are Gamma(a_1, b_1) and Gamma(a_2, b_2) and it has the *pdf*s

$$\pi_1(\beta_1 \mid a_1,b_1) \propto \beta_1^{a_1-1} e^{-b_1\beta_1} \quad \text{if } \beta_1 > 0, \quad (18)$$

and

$$\pi_2(\beta_2 \mid a_2,b_2) \propto \beta_2^{a_2-1} e^{-b_2\beta_2} \quad \text{if } \beta_2 > 0. \quad (19)$$

The gamma parameters a_1, b_1, a_2 and b_2 are all assumed to be positive. When $a_1 = b_1 = 0$, $a_2 = b_2 = 0$, we obtain the non-informative priors of β_1 and β_2 . The joint posterior density of β_1 and β_2 based on the gamma priors is given by

$$\pi^*(\beta_1,\beta_2 \mid \underline{x}) \propto \beta_1^{m_1+a_1-1} \beta_2^{m_2+a_2-1} e^{-\beta_1(b_1-k\,\eta_2(\alpha,\underline{x}))} e^{-\beta_2(b_2-k\,\eta_2(\alpha,\underline{x}))}. \quad (20)$$

It is clear from Eq. (20) that the posterior density functions of β_1 and β_2 say $\pi_1^*(\beta_1 \mid \underline{x})$ and $\pi_2^*(\beta_2 \mid \underline{x})$, respectively, are independent. Further, $\pi_1^*(\beta_1 \mid \underline{x})$ is a gamma density with shape parameter ($m_1 + a_1$) and scale parameter $(b_1 - k\,\eta_2(\alpha,\underline{x}))$ and $\pi_2^*(\beta_2 \mid \underline{x})$ is the gamma density with shape parameter $(m_2 + a_2)$ and scale parameter $(b_2 - k\,\eta_2(\alpha,\underline{x}))$. Therefore, the Bayes estimates of β_1 and β_2 under SEL functions are

$$\tilde{\beta}_1 = \frac{m_1+a_1}{b_1 - k\,\eta_2(\alpha,\underline{x})} \quad \text{and} \quad \tilde{\beta}_2 = \frac{m_2+a_2}{b_2 - k\,\eta_2(\alpha,\underline{x})}, \quad (21)$$

respectively. Interestingly, for the non-informative priors $a_1 = b_1 = a_2 = b_2 = 0$, the Bayes estimators coincide with the corresponding maximum likelihood estimators.

The credible intervals for β_1 and β_2 can be obtained using the posterior distributions of β_1 and β_2. Note that a posteriori: $Z_1 = 2\beta_1(b_1 - k\,\eta_2(\alpha,\underline{x}))$ and $Z_2 = 2\beta_2(b_2 - k\,\eta_2(\alpha,\underline{x}))$ follow χ^2 distributions with $2(m_1+a_1)$ and $2(m_2+a_2)$ degrees of freedom respectively, provided both $2(m_1+a_1)$ and $2(m_2+a_2)$ are positive integers. Therefore, $100(1-\gamma)\%$ credible intervals for β_1 and β_2 are

$$\left(\frac{\chi^2_{2(m_1+a_1),1-\frac{\gamma}{2}}}{2(b_1 - k\,\eta_2(\alpha,\underline{x}))}, \frac{\chi^2_{2(m_1+a_1),\frac{\gamma}{2}}}{2(b_1 - k\,\eta_2(\alpha,\underline{x}))}\right) \text{and} \left(\frac{\chi^2_{2(m_2+a_2),1-\frac{\gamma}{2}}}{2(b_2 - k\,\eta_2(\alpha,\underline{x}))}, \frac{\chi^2_{2(m_2+a_2),\frac{\gamma}{2}}}{2(b_2 - k\,\eta_2(\alpha,\underline{x}))}\right),$$

respectively for $(m_1+a_1) > 0$ and $(m_2+a_2) > 0$. Note that if $2(m_1+a_1)$ and $2(m_2+a_2)$ are not integer values then gamma distribution can be used to construct the credible intervals. If no prior information is available, then non-informative priors can be used to compute the credible intervals for β_1 and β_2.

4.2 Bayesian Estimations When α is UnKnown

In this subsection, we provide the Bayes estimators of the unknown parameters and the corresponding credible intervals when the common shape parameter α is unknown. In some situations where we do not have sufficient prior information, we can use non-informative prior distribution. This is particularly true for our study. For example, the non-informative uniform prior distribution can be used for parameters α, β_1 and β_2. The joint posterior density will then be in proportion to the likelihood function. Here we consider the more important case when the common shape parameter α is unknown and has the gamma prior with the *pdf*

$$\pi_3(\alpha \mid a_3,b_3) \propto \frac{b_3^{a_3}}{\Gamma(a_3)} \alpha^{a_3-1} e^{-b_3\alpha} \quad \text{if } \alpha > 0. \quad (22)$$

Combining the likelihood function (5) with the prior distributions, the joint posterior distribution for α, β_1 and β_2 given data becomes:

$$\pi^*_{\alpha,\beta_1,\beta_2}(\alpha,\beta_1,\beta_2 \mid \underline{x}) = \frac{\ell(\underline{x}|\alpha,\beta_1,\beta_2)\times\pi_1(\beta_1|a_1,b_1)\times\pi_2(\beta_2|a_2,b_2)\times\pi_3(\alpha|a_3,b_3).}{\int_0^\infty\int_0^\infty\int_0^\infty \ell(\underline{x}|\alpha,\beta_1,\beta_2)\times\pi_1(\beta_1|a_1,b_1)\times\pi_2(\beta_2|a_2,b_2)\times\pi_3(\alpha|a_3,b_3)\,d\alpha\, d\beta_1 d\beta_2}. \quad (23)$$

Therefore, the Bayes estimate of any function of α , β_1 and β_2 say $g(\alpha,\beta_1,\beta_2)$, under SEL function is

$$\tilde{g}(\alpha,\beta_1,\beta_2) = E_{\alpha,\beta_1,\beta_2|\underline{x}}(g(\alpha,\beta_1,\beta_2))$$

$$= \frac{\int_0^\infty\int_0^\infty\int_0^\infty g(\alpha,\beta_1,\beta_2)\ell(\underline{x}|\alpha,\beta_1,\beta_2)\times\pi_1(\alpha|a_1,b_1)\times\pi_2(\beta_1|a_2,b_2)\pi_3(\beta_2|a_3,b_3)\,d\alpha\, d\beta_1 d\beta_2}{\int_0^\infty\int_0^\infty\int_0^\infty \ell(\underline{x}|\alpha,\beta_1,\beta_2)\times\pi_1(\alpha|a_1,b_1)\times\pi_2(\beta_1|a_2,b_2)\pi_3(\beta_2|a_3,b_3)\,d\alpha\, d\beta_1 d\beta_2}. \quad (24)$$

It is not possible to compute (24) analytically. Therefore, we propose the MCMC method to approximate (24). In this area we consider the MCMC method to generate samples from the posterior distributions and then compute the Bayes estimates of α, β_1 and β_2 under progressively first-failure censored competing risks data from Kumaraswamy distribution. A wide variety of MCMC schemes are available, and it can be difficult to choose among them. An important sub-class of MCMC methods are Gibbs sampling and more general Metropolis-within-Gibbs samplers; see, for example, [21].

In this section, we propose Gibbs sampling procedure to generate a sample from the posterior density function $\pi^*_{\alpha,\beta_1,\beta_2}(\alpha,\beta_1,\beta_2 \mid \underline{x})$ and in turn compute the Bayes estimates and also construct the corresponding credible intervals based on the generated posterior sample. In order to use the method of MCMC for estimating the parameters of the Kumaraswamy distribution, namely, α, β_1 and β_2. Let us consider independent priors (18), (19) and (22) for the parameters β_1, β_2 and α, respectively. The expression for the posterior can be obtained up to proportionality by multiplying the likelihood with the prior and this can be written as

where $\eta_1(\alpha,\underline{X})$ and $\eta_2(\alpha,\underline{x})$ are defined in Eq. (17). The posterior is obviously complicated and no closed form inferences appear possible. We, therefore, propose to consider MCMC methods, namely the Gibbs sampler, to simulate samples from the posterior so that sample-based inferences can be easily drawn. From (25), the full conditional posterior density of α is proportional to

$$\pi^*_{\alpha}(\alpha \mid \beta_1, \beta_2, \underline{x}) \propto \alpha^{m+a_3-1}\eta_1(\underline{x},\alpha)\exp\left[k(\beta_1+\beta_2)\eta_2(\alpha,\underline{x})-b_3\alpha\right]. \quad (26)$$

Similarly, the full conditional posterior conditional distribution for β_1 and β_2 as the following

$$\pi^*_{\beta_1}(\beta_1 \mid \alpha, \beta_2, \underline{x}) \propto \beta_1^{m_1+a_1-1}\exp[-\beta_1(b_1-k\,\eta_2(\alpha,\underline{x}))], \quad (27)$$

It can be seen that Eq. (27) is a gamma density with shape parameter (m_1+a_1) and scale parameter $(b_1-k\,\eta_2(\alpha,\underline{x}))$, Eq. (20) is a gamma density with shape parameter (m_2+a_2) and scale parameter $(b_2-k\,\eta_2(\alpha,\underline{x}))$ and, therefore, samples of β_1 and β_2 can be easily generated using any gamma generating routine. However, in our case, the conditional posterior distribution of α given β_1 and β_2 Eq. (26) cannot be reduced analytically to well known distributions and therefore it is not possible to sample directly by standard methods. In this case, we need to use the M--H algorithm to generate from the distributions of α. We propose the following MCMC procedure.

MCMC Algorithm:

Step 1: Start with some initial guess of α , say $\alpha^{(0)}$.
Step 2: Set $t=1$.
Step 3: Generate $\beta_1^{(t)}$ from Gamma $(m_1+a_1, b_1-k\,\eta_2(\alpha^{(t-1)},\underline{x}))$.
Step 4: Generate $\beta_2^{(t)}$ from Gamma $(m_2+a_2, b_2-k\,\eta_2(\alpha^{(t-1)},\underline{x}))$.
Step 5: Using Metropolis-Hastings (see, [22]), with a target distribution $\pi^*_{\alpha}(\alpha \mid \beta_1^{(t)}, \beta_2^{(t)}, \underline{x})$ generate $\alpha^{(t)}$ with the proposal distribution N $(\alpha^{(t-1)},1)$.
Step 6: Compute $\alpha^{(t)}$, $\beta_1^{(t)}$ and $\beta_2^{(t)}$.
Step 7: Set $t=t+1$.
Step 8: Repeat steps 3-6 N times representing $\left(\phi_t^{[1]},\phi_t^{[2]},\ldots,\phi_t^{[N]}\right)$, $t=1,2,3$ where ($\phi_1=\alpha$, $\phi_2=\beta_1$ and $\phi_3=\beta_2$).
Step 9: Obtain the Bayes estimates with respect to the SEL function as

$$\tilde{\phi}_t=\frac{1}{N-M}\sum_{i=M+1}^{N}\phi_t^{[i]},\ t=1,2,3 \quad (29)$$

where M is burn-in and MSEs

$$\text{MSE}(\phi_t)=\sqrt{\frac{1}{N-M}\sum_{i=M+1}^{N}\left(\phi_t^{[i]}-\tilde{\phi}_t\right)^2},t=1,2,3 \quad (30)$$

Step 10: To compute the credible intervals. Arrange all $\left(\phi_t^{[M]},\phi_t^{[M+1]},\ldots,\phi_t^{[N]}\right)$ in an ascending order to obtain MCMC sample $\left(\phi_t^{(1)},\phi_t^{(2)},\ldots,\phi_t^{(N-M)}\right)$, $t=1,2,3$. Then the $100(1-2\gamma)\%$ symmetric credible intervals ϕ_t given by

$$\left[\phi_t^{(\gamma(N-M))},\ \phi_t^{((1-\gamma(N-M))}\right]. \quad (31)$$

5 Data Analysis

we consider in this section a real-life data set which was originally reported by [17] and latter analyzed by several authors, see for example [23], [3] and [2]. It was obtained from a laboratory experiment in which male mice received a radiation dose of 300 roentgens. The cause of death for each mouse was determined by autopsy. Restricting the analysis to two causes of death, for the purpose of analysis, we consider reticulum cell sarcoma as cause 1 and combine the other causes of death as cause 2. There were $n_1=29$ deaths due to cause 1 and $n_1=39$ deaths due to cause 2. $n=68$ observations remain in the analysis.

The mean, median, standard deviation and the coefficient of skewness for the two causes of death are calculated as (344.034, 259, 170.568, 1.106) and (412.923, 431, 203.518, -0.227), respectively. For computational ease, we have divided each data point by 1000.

Before progressing further we wish to examine the Kolmogorov Smirnov (K-S) statistic whether the Kumaraswamy model is suitable for this data. The maximum likelihood estimates of α and β based on the two causes of death are (1.984, 5.510) and (1.786, 3.474), respectively. In deaths due to cause 1 the K-S distance and the associated p-value are 0.1934 and 0.2282, respectively, and for the deaths due to cause 2 the corresponding values are 0.0875 and 0.9263.. Based on the p-values, the Kumaraswamy model is found to fit the data very well. We have plotted the empirical survival function, and the fitted survival functions in Fig. 1 and Fig. 2 for both data sets. Observe that they fit the data very well.

Now there were $n=68$ observations in the data. The data are randomly grouped into 34 groups with ($k=2$) items withen each group. Now, we suppose that the pre-determined progressively first-failure-censoring scheme is given by ($R_1=R_2=2$, $R_3=R_4=\cdots=R_{12}=1$, $R_{13}=R_{14}=\cdots=R_{20}$), then a progressively first-failure censored competing risks data of size 20 out of 34 groups of death time is obtained as

(0.040,2), (0.042,2), (0.062,2), (0.158,1), (0.179,2), (0.195,1), (0.212,1), (0.222,2), (0.229,1), 0.244,1), (0.252,2), (0.259,1), (0.301,1), (0.333,2), (0.366,2), (0.407,2), (0.431,2), (0.482,2), (0.620,2), (0.761,2).

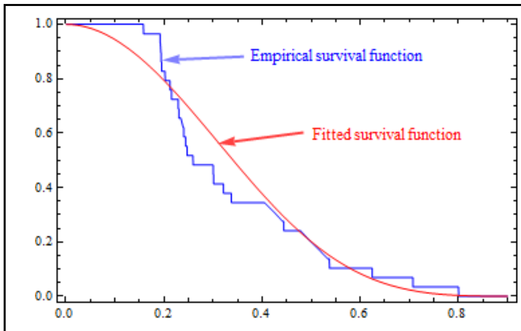

Fig. 1. The empirical and fitted survival functions for deaths due to cause 1.

There were $m_1=7$ deaths due to cause 1 and $m_2=13$ deaths due to cause 2. Based on the above progressively first-failure censored competing risks data, we obtain the ML estimate $(.)_{ML}$ and 95% confidence intervals with the corresponding lengths for parameters. The Bayes point estimates based on the MCMC $(.)_{Bayes\_MCMC}$ and 95% credible intervals with the corresponding lengths for parameters are computed. We run the chain for 10000 times and discard the first 1000 values as `burnin'. We used a non-informative prior distribution ($a=b=c=d=0$). The point estimates for different methods as well as 95% confidence intervals and credible intervals with the corresponding lengths are presented in Table 1.

TABLE 1
POINT ESTIMATES, 95% CONFIDENCE AND CREDIBLE INTERVALS FOR THE PARAMETERS.

Method of Estimation	Parameters	Estimates	Interval	Length
$(.)_{ML}$	α	1.6694	(1.0455, 2.2932)	1.2477
	β_1	0.7705	(0.0064, 1.5347)	1.5282
	β_2	1.4310	(0.2083, 2.6537)	2.4455
$(.)_{Bayes\_MCMC}$	α	1.6179	(1.024, 2.2503)	1.2263
	β_1	0.7544	(0.2324, 1.7296)	1.4973
	β_2	1.4021	(0.5235, 2.9501)	2.4267

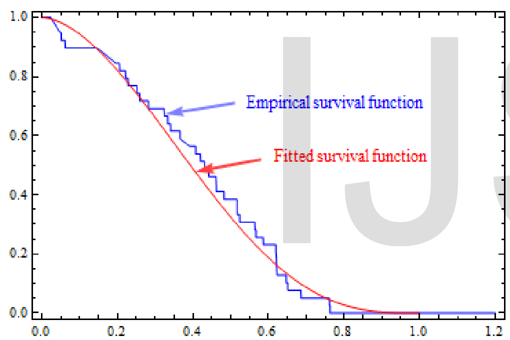

Fig. 1. The empirical and fitted survival functions for deaths due to cause 2.

6 MONTE CARLO SIMULATIONS

In this section, we report some numerical experiments performed to evaluate the behavior of the proposed methods. We simulated 1000 progressively first-failure censored competing risks samples from Kumaraswamy distribution. The samples were generated by using the algorithm described in [24]. We take into consideration that the progressively first-failure censored order statistics $\left(X_{1:m:n:k}^{\mathbf{R}},\delta_1,R_1\right)<\left(X_{2:m:n:k}^{\mathbf{R}},\delta_2,R_2\right)<\cdots<\left(X_{m:m:n:k}^{\mathbf{R}},\delta_m R_m\right)$ is a progressively Type-II censored sample from a population with distribution function $1-(1-F(x))^k$. For each data point, we assigned the cause of failure as 1 or 2 with probability $((\beta_1/(\beta_1+\beta_2))$ and $(\beta_2/(\beta_1+\beta_2))$, respectively. We used different sample sizes n, different effective sample sizes m, differen k and the main patterns of various schemes that are considered in study are given as follows:

$S_{k:m:n}^{(1)}$: All the removals are done at the first failure, i.e. $R_1=n-m$, $R_i=0$ for $i\neq 1$.

$S_{k:m:n}^{(2)}$: The removals are at middle observations, i.e. $R_{\frac{m}{2}}=n-m$, $R_i=0$ for $i\neq\frac{m}{2}$.

$S_{k:m:n}^{(3)}$: The removals are at the last observation, i.e. $R_m=n-m$, $R_i=0$ for $i\neq m$.

We consider two cases separately to draw inference on parameters, namely: (i) known α and (ii) Unknown α

In first case we take $\alpha=3$, $\beta_1=1.5$ and $\beta_2=0.5$ and in the second case we used two sets of parameter values $\alpha=3$, $\beta_1=1$ and $\beta_2=0.5$ and $\alpha=2$, $\beta_1=0.5$ and $\beta_2=0.8$.

For the first case, (known α), very small positive values of a_1,b_1,a_2 and b_2 can be used to construct the Bayes estimates or the corresponding credible intervals. We compute the average Bayes estimates (ABEs) with respect to squared error loss function, mean squared errors (MSEs), average 95% credible interval lengths (ACILs) and the corresponding coverage percentages (CPs). All the results are reported in Tables 2 and 3 .

For the second case (Unknown α) , in this case we consider informative prior for the unknown parameters namely (prior 1: $a_1=3$, $b_1=1$, $a_2=b_2=2$, $a_3=1$, $b_3=2$) and (prior 2: $a_1=4$, $b_1=2$, $a_2=1$, $b_2=2$, $a_3=0.5$, $b_3=0.6$), for the two sets of parameter values. We have chosen the hyper-parameters in such a way that the prior mean became the expected value of the corresponding parameter. We compute the average maximum likelihood estimates (AMEs), mean squared errors (MSEs), average 95% confidence interval lengths (ACILs) and the corresponding coverage percentages (CPs) of the parameters. Also, We compute the average Bayes estimates (ABEs) with respect to squared error loss function, mean squared errors (MSEs), average 95% credible interval lengths (ACILs) and the corresponding coverage percentages (CPs) of the parameters based on 10000 MCMC samples and discard the first 1000 values as burn-in. The results are reported in Tables 2-7.

TABLE 2

THE ABES, ABES OF β_1 AND β_2 AND THEIR MSES (WITHIN BRACKETS) WHEN α IS UNKNOWN, FOR DIFFERENT CENSORING SCHEMES ARE REPORTED. $\beta_1 = 1.5$ AND $\beta_2 = 0.5$.

Scheme	$\bar{\beta}_1$	$\bar{\beta}_2$
$S^{(1)}_{1:20:30}$	1.6044(0.1925)	0.5393(0.0573)
$S^{(2)}_{1:20:30}$	1.5878(0.2081)	0.5041(0.0582)
$S^{(3)}_{1:20:30}$	1.5652(0.2110)	0.5358(0.0649)
$S^{(1)}_{1:30:40}$	1.5551(0.1755)	0.5207(0.0609)
$S^{(2)}_{1:30:40}$	1.6016(0.1953)	0.5407(0.0634)
$S^{(3)}_{1:30:40}$	1.5817(0.1736)	0.5387(0.0551)
$S^{(1)}_{5:20:30}$	1.5726(0.1869)	0.5315(0.0573)
$S^{(2)}_{5:20:30}$	1.5648(0.1879)	0.5434(0.0738)
$S^{(3)}_{5:20:30}$	1.5631(0.1774)	0.5237(0.0533)
$S^{(1)}_{5:30:40}$	1.5410(0.1805)	0.5286(0.0562)
$S^{(2)}_{5:30:40}$	1.5522(0.1654)	0.5217(0.0557)
$S^{(3)}_{5:30:40}$	1.6205(0.2123)	0.5267(0.0588)

TABLE 3

THE 95% ACIL AND THE CORRESPONDING CPS (WITHIN BRACKETS) WHEN α IS KNOWN, FOR DIFFERENT CENSORING SCHEMES ARE REPORTED. $\beta_1 = 1.5$ AND $\beta_2 = 0.5$.

Scheme	$\bar{\beta}_1$	$\bar{\beta}_2$
$S^{(1)}_{1:20:30}$	1.6123(0.958)	0.9053(0.938)
$S^{(2)}_{1:20:30}$	1.5848(0.934)	0.8626(0.924)
$S^{(3)}_{1:20:30}$	1.5769(0.946)	0.8962(0.938)
$S^{(1)}_{1:30:40}$	1.5618(0.948)	0.8726(0.931)
$S^{(2)}_{1:30:40}$	1.6106(0.952)	0.9083(0.928)
$S^{(3)}_{1:30:40}$	1.5926(0.968)	0.9032(0.942)
$S^{(1)}_{5:20:30}$	1.5816(0.946)	0.8928(0.948)
$S^{(2)}_{5:20:30}$	1.5787(0.944)	0.9007(0.928)
$S^{(3)}_{5:20:30}$	1.5704(0.958)	0.8819(0.942)
$S^{(1)}_{5:30:40}$	1.5526(0.944)	0.8815(0.931)
$S^{(2)}_{5:30:40}$	1.5601(0.962)	0.8775(0.944)
$S^{(3)}_{5:30:40}$	1.6223(0.938)	0.8959(0.942)

TABLE 4

THE AMES, ABES OF α, β_1 AND β_2 AND THEIR MSES (WITHIN BRACKETS) WHEN A IS UNKNOWN, FOR DIFFERENT CENSORING SCHEMES ARE REPORTED. $\alpha = 3$, $\beta_1 = 1.0$ AND $\beta_2 = 0.5$.

Scheme	MLE			(Bayes-MCMC)		
	$\hat{\alpha}$	$\hat{\beta}_1$	$\hat{\beta}_2$	$\tilde{\alpha}$	$\tilde{\beta}_1$	$\tilde{\beta}_2$
$S^{(1)}_{1:20:30}$	2.9458	1.0774	0.5075	2.8881	1.0316	0.4891
	(0.3607)	(0.1240)	(0.0595)	(0.2273)	(0.0644)	(0.0323)
$S^{(2)}_{1:20:30}$	3.2455	1.1732	0.5965	3.0709	1.0637	0.5269
	(0.5263)	(0.1986)	(0.1781)	(0.2333)	(0.0755)	(0.0571)
$S^{(3)}_{1:20:30}$	3.0992	1.1273	0.5561	2.9542	1.0487	0.5111
	(0.3801)	(0.1114)	(0.0782)	(0.1738)	(0.0399)	(0.0363)
$S^{(1)}_{1:30:40}$	3.1261	1.0575	0.5584	3.0526	1.0150	0.5306
	(0.4142)	(0.1237)	(0.0639)	(0.2403)	(0.0619)	(0.0367)
$S^{(2)}_{1:30:40}$	3.1435	1.0868	0.5396	3.0535	1.0321	0.5134
	(0.3767)	(0.1603)	(0.0541)	(0.2151)	(0.0799)	(0.0296)
$S^{(3)}_{1:30:40}$	3.0189	1.0674	0.5408	2.9394	1.0188	0.5136
	(0.3956)	(0.1381)	(0.0667)	(0.2113)	(0.0614)	(0.0345)
$S^{(1)}_{5:20:30}$	3.1626	1.2078	0.6081	3.0289	1.0696	0.5428
	(0.3571)	(0.2948)	(0.0793)	(0.1554)	(0.0804)	(0.0307)
$S^{(2)}_{5:20:30}$	3.3160	1.4418	0.7416	3.0871	1.1277	0.5643
	(0.4005)	(0.7140)	(0.2884)	(0.0974)	(0.0821)	(0.0485)
$S^{(3)}_{5:20:30}$	3.1977	1.3629	0.6876	2.9895	1.0679	0.5404
	(0.3531)	(0.7201)	(0.2202)	(0.0891)	(0.0767)	(0.0369)
$S^{(1)}_{5:30:40}$	3.1342	1.1597	0.6160	3.0286	1.0674	0.5631
	(0.2458)	(0.1938)	(0.0938)	(0.1231)	(0.0729)	(0.0411)
$S^{(2)}_{5:30:40}$	3.0778	1.2025	0.5818	2.9723	1.0783	0.5164
	(0.2662)	(0.3113)	(0.1268)	(0.1285)	(0.0910)	(0.0395)
$S^{(3)}_{5:30:40}$	3.1168	1.2331	0.6069	2.9881	1.0634	0.5277
	(0.3220)	(0.5274)	(0.1159)	(0.1077)	(0.0789)	(0.0289)

TABLE 5

THE 95% ACIL AND THE CORRESPONDING CPS (WITHIN BRACKETS) WHEN α IS UNKNOWN, FOR DIFFERENT CENSORING SCHEMES ARE REPORTED. $\alpha = 3$, $\beta_1 = 1.0$ AND $\beta_2 = 0.5$.

Scheme	MLE			(Bayes-MCMC)		
	$\hat{\alpha}$	$\hat{\beta}_1$	$\hat{\beta}_2$	$\tilde{\alpha}$	$\tilde{\beta}_1$	$\tilde{\beta}_2$
$S^{(1)}_{1:20:30}$	2.8122	1.4911	0.8985	2.3788	1.2196	0.7623
	(0.924)	(0.973)	(0.943)	(0.986)	(0.967)	(0.981)
$S^{(2)}_{1:20:30}$	2.9527	1.7792	1.1115	2.3990	1.3312	0.8421
	(0.961)	(0.960)	(0.941)	(0.980)	(0.980)	(0.938)
$S^{(3)}_{1:20:30}$	3.0902	1.8541	1.0998	2.4415	1.3823	0.8524
	(0.943)	(0.943)	(0.952)	(0.933)	(0.945)	(0.952)
$S^{(1)}_{1:30:40}$	2.3226	1.2695	0.7954	2.0788	1.0871	0.6936
	(0.953)	(0.955)	(0.953)	(0.964)	(0.962)	(0.956)
$S^{(2)}_{1:30:40}$	2.4399	1.3119	0.8010	2.1306	1.0893	0.6895
	(0.989)	(0.971)	(0.938)	(0.960)	(0.970)	(0.968)
$S^{(3)}_{1:30:40}$	2.494	1.3728	0.8376	2.1593	1.1311	0.7113
	(0.961)	(0.962)	(0.970)	(0.944)	(0.949)	(0.966)
$S^{(1)}_{5:20:30}$	2.3668	2.3401	1.3348	1.8582	1.5631	0.9576
	(0.961)	(0.959)	(0.963)	(0.965)	(0.963)	(0.966)
$S^{(2)}_{5:20:30}$	2.3994	2.7066	1.7885	1.7956	1.7057	1.0222
	(0.956)	(0.967)	(0.961)	(0.954)	(0.953)	(0.961)
$S^{(3)}_{5:20:30}$	2.7003	3.195	1.9081	1.8473	1.7426	1.1296
	(0.970)	(0.974)	(0.946)	(0.972)	(0.976)	(0.966)
$S^{(1)}_{5:30:40}$	1.9642	1.8407	1.106	1.6191	1.3747	0.8547
	(0.960)	(0.972)	(0.968)	(0.962)	(0.958)	(0.968)
$S^{(2)}_{5:30:40}$	1.8936	2.009	1.1074	1.5497	1.4202	0.8146
	(0.936)	(0.944)	(0.935)	(0.949)	(0.975)	(0.968)
$S^{(3)}_{5:30:40}$	2.1436	2.4019	1.2928	1.6498	1.5127	0.8762
	(0.973)	(0.965)	(0.962)	(0.955)	(0.971)	(0.966)

TABLE 6

THE AMES, ABES OF α, β_1 AND β_2 AND THEIR MSES (WITHIN BRACKETS) WHEN A IS UNKNOWN, FOR DIFFERENT CENSORING SCHEMES ARE REPORTED. $\alpha=2$, $\beta_1=0.5$ AND $\beta_2=0.8$.

Scheme	MLE			(Bayes-MCMC)		
	$\hat{\alpha}$	$\hat{\beta}_1$	$\hat{\beta}_2$	$\tilde{\alpha}$	$\tilde{\beta}_1$	$\tilde{\beta}_2$
$S^{(1)}_{1:20:30}$	2.0091	0.5390	0.8268	1.9661	0.5161	0.8056
	(0.2317)	(0.0467)	(0.0749)	(0.1228)	(0.0272	(0.0523)
$S^{(2)}_{1:20:30}$	2.0950	0.6035	0.8912	2.0007	0.5523	0.8399
	(0.3817)	(0.0997)	(0.1249)	(0.1908)	(0.0407)	(0.0708)
$S^{(3)}_{1:20:30}$	2.3190	0.6975	1.1410	2.1388	0.5982	0.9962
	(0.4645)	(0.1436)	(0.4977)	(0.1642)	(0.0496)	(0.2011)
$S^{(1)}_{1:30:40}$	2.0367	0.5089	0.8666	2.0006	0.4959	0.8481
	(0.2124)	(0.0384)	(0.0739)	(0.1307)	(0.0233)	(0.0541)
$S^{(2)}_{1:30:40}$	2.1728	0.5694	0.9222	2.0922	0.5418	0.8745
	(0.3431)	(0.0352)	(0.1935)	(0.1870)	(0.0204)	(0.0961)
$S^{(3)}_{1:30:40}$	2.2445	0.5832	1.0222	2.1389	0.5351	0.9541
	(0.3558)	(0.0997)	(0.2195)	(0.1697)	(0.0387)	(0.1059)
$S^{(1)}_{5:20:30}$	2.1626	0.7415	1.0139	2.0571	0.6007	0.9103
	(0.1772)	(0.4320)	(0.2209)	(0.0672)	(0.0725)	(0.0742)
$S^{(2)}_{5:20:30}$	2.2377	0.7735	1.2147	2.0932	0.5907	0.9819
	(0.3009)	(0.4587)	(0.7824)	(0.1051)	(0.0558)	(0.1914)
$S^{(3)}_{5:20:30}$	2.1956	0.6884	1.1358	2.0612	0.5593	0.9420
	(0.2078)	(0.1794)	(0.6297)	(0.0649)	(0.0283)	(0.1321)
$S^{(1)}_{5:30:40}$	2.1071	0.6273	0.9738	2.0438	0.5672	0.9049
	(0.1921)	(0.1193)	(0.2491)	(0.1074)	(0.0491)	(0.1211)
$S^{(2)}_{5:30:40}$	2.0916	0.5976	0.9273	2.0391	0.5619	0.8868
	(0.1081)	(0.0603)	(0.1588)	(0.0648)	(0.0306)	(0.0969)
$S^{(3)}_{5:30:40}$	2.1703	0.7059	1.0869	2.0622	0.5981	0.9584
	(0.1755)	(0.1927)	(0.3362)	(0.0703)	(0.0496)	(0.1081)

TABLE 7

THE 95% ACIL AND THE CORRESPONDING CPS (WITHIN BRACKETS) WHEN α IS UNKNOWN, FOR DIFFERENT CENSORING SCHEMES ARE REPORTED. $\alpha=2$, $\beta_1=0.5$ AND $\beta_2=0.8$.

Scheme	MLE			(Bayes-MCMC)		
	$\hat{\alpha}$	$\hat{\beta}_1$	$\hat{\beta}_2$	$\tilde{\alpha}$	$\tilde{\beta}_1$	$\tilde{\beta}_2$
$S^{(1)}_{1:20:30}$	1.9912	0.8756	1.1596	1.6924	0.7508	1.052
	(0.948)	(0.944)	(0.927)	(0.956)	(0.960)	(0.963)
$S^{(2)}_{1:20:30}$	1.9464	1.0268	1.3474	1.6451	0.8212	1.1591
	(0.896)	(0.948)	(0.929)	(0.943)	(0.968)	(0.966)
$S^{(3)}_{1:20:30}$	2.2874	1.3433	1.9862	1.8123	0.9597	1.4816
	(0.971)	(0.967)	(0.967)	(0.973)	(0.970)	(0.967)
$S^{(1)}_{1:30:40}$	1.6794	0.6911	0.9804	1.4876	0.6196	0.9048
	(0.962)	(0.943)	(0.963)	(0.967)	(0.953)	(0.963)
$S^{(2)}_{1:30:40}$	1.7059	0.795	1.1214	1.5093	0.6928	0.9978
	(0.932)	(0.931)	(0.948)	(0.950)	(0.948)	(0.935)
$S^{(3)}_{1:30:40}$	1.8608	0.8891	1.3355	1.6010	0.7259	1.1363
	(0.947)	(0.962)	(0.946)	(0.937)	(0.962)	(0.960)
$S^{(1)}_{5:20:30}$	1.6357	1.5890	1.9957	1.3385	1.0268	1.5162
	(0.942)	(0.940)	(0.953)	(0.944)	(0.963)	(0.954)
$S^{(2)}_{5:20:30}$	1.6258	1.8112	2.6142	1.2986	1.0543	1.6822
	(0.937)	(0.964)	(0.963)	(0.943)	(0.962)	(0.954)
$S^{(3)}_{5:20:30}$	1.8738	1.8412	2.8941	1.3945	1.0817	1.7942
	(0.971)	(0.9763)	(0.966)	(0.961)	(0.966)	(0.966)
$S^{(1)}_{5:30:40}$	1.3527	1.0948	1.5671	1.1716	0.8496	1.2919
	(0.960)	(0.932)	(0.968)	(0.962)	(0.948)	(0.945)
$S^{(2)}_{5:30:40}$	1.2908	1.0465	1.5009	1.1324	0.8543	1.2936
	(0.961)	(0.965)	(0.945)	(0.952)	(0.954)	(0.963)
$S^{(3)}_{5:30:40}$	1.4991	1.4631	2.1173	1.2051	0.9781	1.5386
	(0.962)	(0.973)	(0.953)	(0.955)	(0.962)	(0.970)

7 CONCLUSIONS

In this paper, we have analyzed progressive first-failure-censoring in the presence of competing risks. In particular, we have assumed that the latent failure times under the competing risks follow independent Kumaraswamy distributions with the same one shape parameter and different other shape parameters. The maximum likelihood and Bayes methods are utilized to estimation the model parameters. Additionally, the two-sided confidence and credible interval lengths are computed. When the common shape parameter is known, the Bayes estimates of the other shape parameters have closed form expressions, but when the common shape parameter is unknown, the Bayes estimates do not have explicit expressions. In this case we propose to use MCMC samples to compute the Bayes estimates and the corresponding credible intervals. Based on the results of the simulation study some of the points are clear from this experiment. We observe the following:

i) The results obtained in this paper can be specialized to: (a) first-failure-censored order statistics by taking $R=(0,\cdots,0)$. (b) progressively Type-II censored statistics for $k=1$. (c) usually Type-II censored order statistics for $k=1$ and $R=(0,...,n-m)$. (d) complete sample for $k=1$, $n=m$ and $R=(0,\cdots,0)$.
ii) When the effective sample proportion ($n:m$) increases, the MSEs and the average probability interval lengths of parameters almost decrease in most cases. Also, the CPs in most cases are closed to the nominal level 0.95, (see Tables 2-7).
iii) The MSEs and ACILs for the estimates of the parameters and for the proposed progressively first-failure censored competing risks ($k=5$) are similar to those for progressively Type-II censored competing risks ($k=1$).
iv) The censoring scheme $S^{(1)}_{k:m:n}$ namely, ($R=(n-m,...,0)$, in the sense for fixed n and m, $n-m$ items are removed at the time of the first failure) is most efficient for all choices, it seems to usually provide the smallest MSEs for all estimators.
v) From the results obtained in Tables 2-7. It can be seen that the the Bayes estimators perform better than the MLEs, in terms of both MSEs and the average lengths of the credible intervals.

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